

# CONTINUOUS CM-REGULARITY OF SEMIHOMOGENEOUS VECTOR BUNDLES

ALEX KÜRONYA, YUSUF MUSTOPA

**ABSTRACT.** We show that if  $X$  is an abelian variety of dimension  $g \geq 1$  and  $\mathcal{E}$  is an  $M$ -regular coherent sheaf on  $X$ , the Castelnuovo-Mumford regularity of  $\mathcal{E}$  with respect to an ample and globally generated line bundle  $\mathcal{O}(1)$  on  $X$  is at most  $g$ , and that equality is attained when  $\mathcal{E}^\vee(1)$  is continuously globally generated. As an application, we give a numerical characterization of ample semihomogeneous vector bundles for which this bound is attained.

## INTRODUCTION

If  $X$  is a smooth projective variety and  $\mathcal{O}(1)$  is an ample and globally generated line bundle on  $X$ , the Castelnuovo-Mumford regularity of  $\mathcal{F}$  with respect to the polarization  $\mathcal{O}(1)$  is defined as

$$\mathrm{reg}(\mathcal{F}, \mathcal{O}(1)) \stackrel{\mathrm{def}}{=} \min \{m \in \mathbb{Z} \mid \forall i > 0 \ H^i(\mathcal{F}(m-i)) = 0\}.$$

This invariant is instrumental in measuring the homological complexity of the graded module  $H_*^0(\mathcal{F}) = \bigoplus_m H^0(\mathcal{F}(m))$ , and at the same time helps measure positivity, since  $\mathcal{F}(m)$  is globally generated for all  $m \geq \mathrm{reg}(\mathcal{F}, \mathcal{O}(1))$ .

In light of Kollár's suggestion that the positivity of adjoint bundles should be controlled by intersections of Chern classes [Kol] we may ask the following: if  $\mathcal{E}$  is an ample vector bundle on  $X$ , when does  $\mathrm{reg}(\omega_X \otimes \mathcal{E}, \mathcal{O}(1))$  depend entirely on the Chern data of  $\mathcal{E}$ ,  $\mathcal{O}(1)$  and  $X$ ? The purpose of this note is to address this question when  $X$  is an abelian variety.

An example of the numerical nature of positivity on abelian varieties can be seen in [KL], where it was shown that the  $N_p$  property for an embedding of an abelian surface  $X$  is "almost numerical" in that its failure to be numerical is detected by an elliptic curve of low degree on  $X$ .

It is well-known that the geometric and homological behaviour of line bundles is considerably tamer on abelian varieties than on an arbitrary variety. A major step in the study of coherent sheaves on abelian varieties was carried out by Pareschi–Popa in [PP1, PP2, PP3], where the theory developed by Mukai in [Mu2] plays an essential role in proving positivity and vanishing statements.

While ampleness of higher-rank vector bundles can be hard to check even on abelian varieties, the property of  $M$ -regularity (see Section 1.2 for the definition) introduced in [PP1] is robust and accessible, and implies ampleness by work of Debarre [De]. Our first result uses CM-regularity to measure the minimal amount of positivity an  $M$ -regular vector bundle can possess.

**Proposition A** (Proposition 2.1). *Let  $X$  be an abelian variety of dimension  $g \geq 2$ ,  $\mathcal{O}(1)$  an ample and globally generated line bundle on  $X$ , and let  $\mathcal{E}$  be an  $M$ -regular coherent sheaf on  $X$ . Then*

$$\mathrm{reg}(\mathcal{E}, \mathcal{O}(1)) \leq g,$$

*with equality if  $\mathcal{E}^\vee(1)$  is  $M$ -regular as well.*

Although  $M$ -regularity is invariant under tensoring by line bundles of degree 0, CM regularity is generally not. A variant of CM-regularity better suited to sheaves on irregular varieties was introduced and studied in [Mus]. The *continuous CM-regularity* of a coherent sheaf  $\mathcal{F}$  with respect to  $\mathcal{O}(1)$ , which is denoted by  $\mathrm{reg}_{\mathrm{cont}}(\mathcal{F}, \mathcal{O}(1))$ , is equal to  $\mathrm{reg}(\mathcal{F} \otimes \alpha, \mathcal{O}(1))$  for general  $\alpha \in \widehat{X}$ ;

see Section 1.2 for details. Proposition A implies that if  $\mathcal{E}$  and  $\mathcal{E}^\vee(1)$  are both  $M$ -regular vector bundles on a  $g$ -dimensional abelian variety, then  $\text{reg}_{\text{cont}}(\mathcal{E}, \mathcal{O}(1)) = \text{reg}(\mathcal{E}, \mathcal{O}(1)) = g$ .

It is natural to ask for a characterization of  $M$ -regular vector bundles with maximal CM-regularity. Here we turn to semihomogeneous bundles, which were first considered by Mukai in [Mu1] and are precisely the vector bundles on an abelian variety which admit an Atiyah-type classification. One of their significant features is that all reasonable positivity properties for vector bundles on abelian varieties, such as ampleness and  $M$ -regularity, are equivalent for them (Proposition 2.3). A numerical criterion for the surjectivity of multiplication maps associated to semihomogeneous bundles was given in Theorem 6.13 of [PP3].

**Theorem B** (Theorem 2.4). *Let  $(X, \mathcal{O}(1))$  be a polarized abelian variety of dimension  $g \geq 1$  with  $\mathcal{O}(1)$  globally generated, and let  $\mathcal{E}$  be an ample semihomogeneous vector bundle on  $X$ . Then  $\text{reg}_{\text{cont}}(\mathcal{E}, \mathcal{O}(1)) = g$  precisely when  $\det(\mathcal{E}^\vee(1))$  is an ample line bundle.*

This result gives a class of vector bundles whose continuous CM-regularity is characterized numerically. Moreover, the role of continuous CM-regularity in the statement cannot be replaced by ordinary CM-regularity, as can be seen from setting  $\mathcal{E} = \mathcal{O}(1)$ .

Our proof of Theorem B uses a generic vanishing result from [PP2] which implies that every nef line bundle on an abelian variety is a generic vanishing sheaf (compare Remark 2.5). For examples of semihomogeneous bundles of arbitrarily high rank satisfying our conditions, see the Verlinde bundles in Remark 2.6.

**Acknowledgements.** This research started while the authors participated in the Combinatorial Algebraic Geometry Major Thematic Program at the Fields Institute; we would like to thank its organizers for the motivating atmosphere. We would also like to thank Mihnea Popa for useful comments on an earlier draft.

## 1. PRELIMINARIES

**1.1. Notation and conventions.** We work over the complex numbers. Throughout,  $X$  is an abelian variety and  $\mathcal{O}(1)$  is an ample and globally generated line bundle on  $X$ .

**1.2. Positivity Notions.** If  $\mathcal{F}$  is a coherent sheaf on  $X$  and  $0 \leq i \leq \dim(X)$ , the  $i$ -th cohomological support locus of  $\mathcal{F}$  is

$$V^i(\mathcal{F}) \stackrel{\text{def}}{=} \{\alpha \in \text{Pic}^0(X) \mid H^i(\mathcal{F} \otimes \alpha) \neq 0\}$$

The following notion was introduced in [Mu2].

**Definition 1.1.**  $\mathcal{F}$  is I.T. of index 0 if  $V^i(\mathcal{F}) = \emptyset$  for all  $i > 0$ .

As is well-known, a line bundle on an abelian variety is ample if and only if it is I.T. of index 0. The following more general notion was introduced and studied in [PP1].

**Definition 1.2.**  $\mathcal{F}$  is  $M$ -regular if  $\text{codim}(V^i(\mathcal{F})) > i$  for all  $i > 0$ .

Much of the power of this definition comes from [PP1, Proposition 2.13], which says that  $M$ -regular sheaves satisfy the following property.

**Definition 1.3.**  $\mathcal{F}$  is *continuously globally generated* if there is a nonempty Zariski-open subset  $U \subseteq \text{Pic}^0(X)$  such that the evaluation map

$$(1.1) \quad \bigoplus_{\alpha \in U} H^0(\mathcal{F} \otimes \alpha) \otimes \alpha^\vee \rightarrow \mathcal{F}$$

is surjective.

While this is not strictly comparable to global generation, it is closely related. The next result, which will be used in our proof of Proposition A, is a consequence of the proof of [De, Proposition 3.1].

**Proposition 1.4.** *Let  $\mathcal{E}$  be a coherent sheaf on an abelian variety  $X$ . Then the following are equivalent:*

- (i)  $\mathcal{E}$  is continuously globally generated.
- (ii) *There exists an isogeny  $\pi : Y \rightarrow X$  of abelian varieties such that  $\pi^*(\mathcal{E} \otimes \alpha)$  is globally generated for all  $\alpha \in \widehat{X}$ .*  $\square$

By [De] this implies that every continuously globally generated vector bundle on  $X$ , and therefore any  $M$ -regular vector bundle on  $X$ , is ample. Our proof of Proposition A also requires the following result from [PP1].

**Proposition 1.5.** [PP1, Proposition 2.9] *If  $\mathcal{F}$  is  $M$ -regular and  $\mathcal{H}$  is a locally free sheaf on  $X$  which is I.T. of index 0, then  $\mathcal{F} \otimes \mathcal{H}$  is I.T. of index 0.*  $\square$

We now come to continuous Castelnuovo-Mumford regularity; for its basic properties we refer the reader to [Mus]. A related notion, that of *continuous rank*, has recently been studied in [BPS].

**Definition 1.6.** The *continuous CM-regularity* of  $\mathcal{F}$  with respect to  $\mathcal{O}(1)$  is

$$\text{reg}_{\text{cont}}(\mathcal{F}, \mathcal{O}(1)) \stackrel{\text{def}}{=} \min \{m \in \mathbb{Z} \mid \forall i > 0 \ V^i(\mathcal{F}(m-i)) \neq \text{Pic}^0(X)\}$$

## 2. PROOFS

**Proposition 2.1.** *Let  $X$  be an abelian variety of dimension  $g \geq 2$ , let  $\mathcal{O}(1)$  be an ample and globally generated line bundle on  $X$ , and let  $\mathcal{E}$  be a coherent sheaf on  $X$  which is  $M$ -regular. Then we have the following:*

- (i)  $\text{reg}_{\text{cont}}(\mathcal{E}, \mathcal{O}(1)) \leq \text{reg}(\mathcal{E}, \mathcal{O}(1)) \leq g$ .
- (ii) *If  $\mathcal{E}$  is locally free and  $\mathcal{E}^\vee(1)$  is continuously globally generated, then*

$$\text{reg}_{\text{cont}}(\mathcal{E}, \mathcal{O}(1)) = \text{reg}(\mathcal{E}, \mathcal{O}(1)) = g.$$

*Proof.* We first establish (i). Let  $\alpha \in \widehat{X}$  be given. Since  $\mathcal{O}(1)$  is ample,  $\mathcal{O}(g-i)$  is I.T. of index 0 for  $1 \leq i \leq g-1$ . Given that  $\mathcal{E}$  is  $M$ -regular, Proposition 1.5 implies that  $\mathcal{E}(g-i) \otimes \alpha$  is I.T. of index 0 for  $1 \leq i \leq g-1$ . Also, the  $M$ -regularity of  $\mathcal{E}$  implies that the cohomological support locus  $V^g(\mathcal{E})$  is empty, so that  $H^g(\mathcal{E} \otimes \alpha) = 0$ . Therefore  $\text{reg}(\mathcal{E} \otimes \alpha, \mathcal{O}(1)) \leq g$  for all  $\alpha \in \widehat{X}$ . Setting  $\alpha = \mathcal{O}_X$ , we have that  $\text{reg}(\mathcal{E}, \mathcal{O}(1)) \leq g$ . This concludes the proof of (i) since  $\text{reg}_{\text{cont}}(\mathcal{E}, \mathcal{O}(1)) \leq \text{reg}(\mathcal{E}, \mathcal{O}(1))$  by definition.

Turning to (ii), it is enough to show that  $H^g(\mathcal{E}(-1) \otimes \alpha) \neq 0$  for all  $\alpha \in \widehat{X}$ . By our hypothesis on  $\mathcal{E}^\vee(1)$ , Proposition 1.4 implies that there exists an abelian variety  $Y$  of dimension  $g$  and an isogeny  $\pi : Y \rightarrow X$  such that  $\pi^*(\mathcal{E}^\vee(1) \otimes \alpha)$  is globally generated for all  $\alpha \in \widehat{X}$ . In particular, we have that

$$(2.1) \quad \bigoplus_{\xi \in \ker(\pi^*)} H^0(\mathcal{E}^\vee(1) \otimes (\alpha \otimes \xi)) = H^0(\mathcal{E}^\vee(1) \otimes \alpha \otimes \pi_* \mathcal{O}_Y) \cong H^0(\pi^*(\mathcal{E}^\vee(1) \otimes \alpha)) \neq 0$$

for all  $\alpha \in \widehat{X}$ . Since  $\ker(\pi^*)$  is finite and the cohomological support loci of coherent sheaves on  $X$  are algebraic subsets of  $\widehat{X}$ , there exists  $\xi' \in \ker(\pi^*)$  such that

$$(2.2) \quad H^g(\mathcal{E}(-1) \otimes \alpha \otimes \xi'^\vee) \cong H^0(\mathcal{E}^\vee(1) \otimes \alpha^\vee \otimes \xi')^* \neq 0$$

for all  $\alpha \in \widehat{X}$ .  $\square$

**Remark 2.2.** In [De] it is shown that given a branched covering  $f : Y \rightarrow X$  of a simple abelian variety which does not factor through a nontrivial isogeny, the associated Tschirnhausen bundle  $E_f$  is  $M$ -regular. It can be checked that if  $h^0(\mathcal{O}_Y(1) \otimes f^* \alpha^\vee) > h^0(\mathcal{O}_X(1) \otimes \alpha)$  for all  $\alpha \in \widehat{X}$ , we have  $\text{reg}_{\text{cont}}(E_f, \mathcal{O}(1)) = g$ .

Next, we proceed to the case of semihomogeneous vector bundles.

**Proposition 2.3.** *Let  $\mathcal{E}$  be a semihomogeneous vector bundle on  $X$ . Then the following are equivalent:*

- (i)  $\mathcal{E}$  is I.T. of index 0.
- (ii)  $\mathcal{E}$  is  $M$ -regular.
- (iii)  $\mathcal{E}$  is continuously globally generated.
- (iv)  $\mathcal{E}$  is ample.
- (v)  $\det(\mathcal{E})$  is ample.

*Proof.* The implication (i)  $\Rightarrow$  (ii) is immediate from the definitions, and (ii)  $\Rightarrow$  (iii) follows from [PP1, Proposition 2.13] while (iii)  $\Rightarrow$  (iv) follows from [De]. An ample vector bundle has an ample determinant, which takes care of (iv)  $\Rightarrow$  (v), and we are left with proving (v)  $\Rightarrow$  (i).

We will see that semihomogeneous bundles with ample determinant are necessarily I.T. of index 0. Indeed, assume first that  $\mathcal{E}$  is simple. By the ampleness of  $\det(\mathcal{E})$  it is non-degenerate as well, therefore it is I. T. of index 0 according to [PP3, Lemma 6.5]. In general,  $\mathcal{E}$  possesses a direct sum decomposition

$$\mathcal{E} \simeq \bigoplus_{i=1}^n \mathcal{E}_i$$

such that all  $\mathcal{E}_i$  have a filtrations with factors isomorphic to a given simple semihomogeneous vector bundle  $\mathcal{F}_i$  for  $1 \leq i \leq n$  (see [Mu1, Proposition 6.18]), and all the  $\mathcal{E}_i$ 's have the same slope as elements of  $N^1(X)_{\mathbb{Q}}$ . Therefore,  $\det(\mathcal{E}_i)$  must be ample for all  $1 \leq i \leq n$ . But then  $\mathcal{F}_i$  is ample as a quotient of  $\mathcal{E}_i$ , hence I. T. of index 0. A short cohomology computation shows that  $\mathcal{E}_i$  must be I. T. of index 0 as well, hence so is  $\mathcal{E}$ , as required.  $\square$

**Theorem 2.4.** *Let  $X$  be an abelian variety of dimension  $g \geq 1$ , let  $\mathcal{O}(1)$  be an ample and globally generated line bundle on  $X$ , and let  $\mathcal{E}$  be an indecomposable semihomogeneous vector bundle of rank  $r \geq 1$  on  $X$  which is ample. Then the following are equivalent.*

- (i)  $\text{reg}_{\text{cont}}(\mathcal{E}, \mathcal{O}(1)) = g$ .
- (ii)  $\det(\mathcal{E}^\vee(1))$  is an ample line bundle.

*Proof.* (i)  $\Rightarrow$  (ii): Arguing as in the proof of Proposition 2.1, we may assume without loss of generality that  $\mathcal{E}$  is simple. Then there exists an isogeny  $\pi : X' \rightarrow X$  and a line bundle  $\mathcal{L}$  on  $X'$  such that  $\mathcal{E} \cong \pi_* \mathcal{L}$ . For all  $\eta \in \widehat{X}$  and all  $i \geq 0$  we then have

$$(2.3) \quad H^i(\mathcal{E}(-1) \otimes \eta) \cong H^i(\mathcal{L} \otimes \pi^* \mathcal{O}(-1) \otimes \pi^* \eta)$$

Since  $\text{reg}_{\text{cont}}(\mathcal{E}, \mathcal{O}(1)) = g$  by hypothesis and  $\mathcal{E}(g-1-i)$  is  $M$ -regular for  $1 \leq i \leq g-1$ , we must have  $V^g(\mathcal{E}(-1)) = \widehat{X}$ . The dual isogeny  $\pi^* : \widehat{X} \rightarrow \widehat{X'}$  is surjective, so it follows from (2.3) that

$$(2.4) \quad V^0(\mathcal{L}^\vee \otimes \pi^* \mathcal{O}(1)) = -V^g(\mathcal{L} \otimes \pi^* \mathcal{O}(-1)) = -(\pi^*)(V^g(\mathcal{E}(-1))) = \widehat{X'}$$

Consequently  $\mathcal{L}^\vee \otimes \pi^* \mathcal{O}(1)$  is an effective line bundle on an abelian variety, and therefore nef. We will be done once we verify that  $\mathcal{L}^\vee \otimes \pi^* \mathcal{O}(1)$  is nondegenerate of index 0. Indeed, this implies via (2.3) and Serre duality that  $\mathcal{E}^\vee(1)$  is nondegenerate of index 0. Since  $\mathcal{E}^\vee(1)$  is semihomogeneous, this implies the ampleness of  $\det(\mathcal{E}^\vee(1))$  by Proposition 6.2 of [Gul].

By Corollary C of [PP2], any nef line bundle on  $X'$  is a generic vanishing sheaf, so for  $i > 0$  and general  $\eta' \in \widehat{X'}$  we have that  $H^i(\mathcal{L}^\vee \otimes \pi^* \mathcal{O}(1) \otimes \eta') = 0$ . Therefore for all such  $i$  and  $\eta'$ , we have that

$$(2.5) \quad \chi(\mathcal{L}^\vee \otimes \pi^* \mathcal{O}(1)) = \chi(\mathcal{L}^\vee \otimes \pi^* \mathcal{O}(1) \otimes \eta') = h^0(\mathcal{L}^\vee \otimes \pi^* \mathcal{O}(1) \otimes \eta') > 0$$

so that  $\mathcal{L}^\vee \otimes \pi^* \mathcal{O}(1)$  is nondegenerate of index 0 as desired.

(ii)  $\Rightarrow$  (i): Since  $\det \mathcal{E}^\vee(1)$  is ample by hypothesis,  $\mathcal{E}^\vee(1)$  is continuously globally generated by Proposition 2.3. Applying (ii) of Proposition 2.1 finishes the proof.  $\square$

**Remark 2.5.** By Corollary 5.4 of [Gul] a line bundle on an abelian variety is nef if and only if it is a numerically trivial twist of the pullback of an ample line bundle from a quotient abelian variety. While we appeal to Corollary C of [PP2] for the sake of brevity, one can also use an *ad hoc* argument using a Leray spectral sequence associated to the quotient map.

**Remark 2.6.** For every  $g \geq 1$  there exists an abelian variety  $X$  of dimension  $g$ , an ample and globally generated line bundle  $\mathcal{O}(1)$  on  $X$ , and an ample semihomogeneous bundle on  $X$  of rank  $\geq g$  such that  $\mathcal{E}$  and  $\mathcal{O}(1)$  satisfy the equivalent conditions of Theorem 2.4; these are examples of Verlinde bundles, which were studied in [O]. It follows from Theorem 7.11 and Remark 7.13 of [Mu1] that when  $(X, \Theta)$  is a principally polarized abelian variety of dimension  $g$  and  $a$  and  $b$  are odd coprime positive integers, there exists a simple semihomogeneous bundle  $\mathcal{E}$  on  $X$  satisfying

$$(2.6) \quad \text{rank}(\mathcal{E}) = a^g, \quad \det(\mathcal{E}) \cong \Theta^{a^{g-1}b}$$

In particular,  $\mathcal{E}$  is ample, and we may take any  $\mathcal{O}(1)$  such that  $\mathcal{O}(a^g)(-a^{g-1}b\Theta)$  is ample.

We end with an example illustrating what can happen when the continuous CM-regularity is not maximal. In what follows,  $X = E_1 \times E_2$ , where  $E_1$  and  $E_2$  are non-isogenous elliptic curves. Here the Picard number of  $X$  is 2 and the nef cone of  $X$  is generated by the classes of the fibers  $F_1$  and  $F_2$  of the projection maps.

**Remark 2.7.** Theorem 2.4 does not extend naively to the case of sub-maximal continuous CM-regularity. Let  $p_i, q_i$  be distinct points in  $E_i$  for  $i = 1, 2$ . Define  $\mathcal{O}(1) := \mathcal{O}_{E_1}(2p_1) \boxtimes \mathcal{O}_{E_2}(2p_2)$  and  $\mathcal{L} := \mathcal{O}_{E_1}(5q_1) \boxtimes \mathcal{O}_{E_2}(q_2)$ . We claim that  $\text{reg}_{\text{cont}}(\mathcal{L}, \mathcal{O}(1)) = 1$  even though  $\mathcal{L}^\vee(2) \cong \mathcal{O}_{E_1}(4p_1 - 5q_1) \boxtimes \mathcal{O}_{E_2}(4p_2 - q_2)$  is not ample.

To see that  $\text{reg}_{\text{cont}}(\mathcal{L}, \mathcal{O}(1)) \leq 1$ , observe that for all  $\alpha_i \in \widehat{E}_i, i = 1, 2$  we have  $h^1(\mathcal{L} \otimes (\alpha_1 \boxtimes \alpha_2)) = 0$  due to the ampleness of  $\mathcal{L}$  and also

$$(2.7) \quad h^2(\mathcal{L}(-1) \otimes (\alpha_1 \boxtimes \alpha_2)) = h^1(\mathcal{O}_{E_1}(5q_1 - 2p_1) \otimes \alpha_1) \cdot h^1(\mathcal{O}_{E_2}(q_2 - 2p_2) \otimes \alpha_2) = 0$$

To see that  $\text{reg}_{\text{cont}}(\mathcal{L}, \mathcal{O}(1)) \geq 1$ , observe that for all  $\alpha_i \in \widehat{E}_i, i = 1, 2$  we have

$$(2.8) \quad h^1(\mathcal{L}(-1) \otimes (\alpha_1 \boxtimes \alpha_2)) \geq h^0(\mathcal{O}_{E_1}(5q_1 - 2p_1) \otimes \alpha_1) \cdot h^1(\mathcal{O}_{E_2}(q_2 - 2p_2) \otimes \alpha_2) > 0$$

**Remark 2.8.** The region in  $N_{\mathbb{R}}^1(X)$  on which the continuous CM-regularity of a line bundle is equal to 1 can be somewhat involved. Let  $H = a_1 F_1 + a_2 F_2$  where  $a_1, a_2 \geq 2$  and let  $L = b_1 F_1 + b_2 F_2$  where  $b_1, b_2 \geq 1$ . We already know from Theorem 2.4 (for instance) that the continuous regularity of  $L$  with respect to  $H$  is 2 if and only if  $a_1 > b_1$  and  $a_2 > b_2$ .

For the continuous regularity to be at most 1, we need that  $b_1 \geq a_1$  or  $b_2 \geq a_2$ . For the continuous regularity to be at least 1, we need one of the following to be true:

$$(2.9) \quad b_1 < 2a_1, \text{ and } b_2 < 2a_2$$

$$(2.10) \quad (b_1 > a_1 \text{ and } b_2 < a_2) \text{ or } (b_1 < a_1 \text{ and } b_2 > a_2)$$

These conditions determine a non-convex unbounded polygonal region in the two-dimensional vector space  $N_{\mathbb{R}}^1(X)$ .

#### REFERENCES

- [BPS] M. A. Barja, R. Pardini and L. Stoppino, *Linear Systems on Irregular Varieties*, preprint, [arXiv:1606.03290](#).
- [De] O. Debarre, *On coverings of simple abelian varieties*, Bull. Math. soc. France **134** (2006), no. 2, p. 253-260.
- [Gul] M. Gulbrandsen, *Fourier-Mukai transforms of line bundles on derived equivalent abelian varieties*, Matematiche (Catania) **63** (2008), no. 1, p. 123–137.
- [Kol] J. Kollár, *Singularities of Pairs*, in *Algebraic Geometry, Santa Cruz 1995* (1997), p. 221–287.
- [KL] A. Küronya and V. Lozovanu, *A Reider-type theorem for higher syzygies on abelian surfaces*, preprint, [arxiv.org/1509.08621](#).
- [Mu1] S. Mukai, *Semi-homogeneous vector bundles on an Abelian variety*, J. Math. Kyoto Univ. **18** (1978), no. 2, p. 239-272.
- [Mu2] S. Mukai, *Duality between  $\mathcal{D}(X)$  and  $\mathcal{D}(\widehat{X})$  with its application to Picard sheaves*, Nagoya Math. J. **81** (1981), p. 153–175.
- [Mum] D. Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, No. 5, Published for the Tata Institute of Fundamental Research, Bombay, 1970.
- [Mus] Y. Mustopa, *Castelnuovo-Mumford Regularity and GV-Sheaves on Irregular Varieties*, preprint, [arxiv.org/1607.06550](#).
- [O] D. Oprea, *The Verlinde bundles and the semihomogeneous Wirtinger duality*, J. Reine Angew. Math. **654** (2011), p. 181–217.
- [PP1] G. Pareschi and M. Popa, *Regularity on abelian varieties I*, J. Amer. Math. Soc. **16** (2003), p. 285-302.
- [PP2] G. Pareschi and M. Popa, *GV-sheaves, Fourier-Mukai transform, and Generic Vanishing*, Amer. J. Math. **133** (2011), no.1, p. 235-271.
- [PP3] G. Pareschi and M. Popa, *Regularity on abelian varieties III: relationship with generic vanishing and applications*, in *Grassmannians, Moduli Spaces and Vector Bundles*, Clay Mathematics Proceedings **14** (2011), Amer. Math. Soc., Providence, RI, p. 141-167.